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TRANSFORMATIONS OF TRAJECTORIES ON A SURFACE.

BY JOSEPH LIPKA.

1. Trajectories and their properties. In a recent paper* the author proved five geometric properties which completely characterize the system of ∞^3 trajectories generated by the motion of a particle on any constraining surface under any positional field of force. The purpose of this paper is to study the point transformations on the surface which leave some or all of these properties invariant. Each of these properties together with those preceding it defines a type of systems of ∞^3 curves on the surface, and our problem is to find the nature of the transformations which convert any system of such a type into a system of the same type.† We shall here briefly state these properties, giving the differential equations of the systems of curves defined by them.

Let us consider the surface whose equations are

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

referred to an orthogonal set of parameter curves,‡ so that the element of length has the form

$$ds^2 = Edu^2 + Gdv^2. \tag{1}$$

Property I. If the ∞^1 curves passing through a given point in a given direction have associated with them their orthogonal projections in the tangent plane to the surface at the given point, then the locus of the foci of the osculating parabolas of the associate system is a bicircular quartic with the given point as node and the given direction as tangent line; this tangent line is also one of the asymptotes to the hyperbola which is the inverse of the quartic with respect to the given point.

The most general system of ∞^3 curves on a surface possessing property *I* is defined by a differential equation of the form

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* Motion on a surface for any positional field of force, Proc. Amer. Acad. Arts and Sci., vol. 56, no. 4, pp. 155–182. We shall hereafter refer to this paper by the title "Proceedings."

† For the corresponding problem in the plane, see E. Kasner, The trajectories of dynamics, Trans. Amer. Math. Soc., vol. 7, p. 418.

‡ In "Proceedings" we used an isothermal set of parameter curves, so that some of the equations had a simpler form. But for a discussion of point-transformations we find it necessary to use a more general parameter system.

$$(I) \qquad v''' = A + Bv'' + Cv''^2,$$

where A, B, C are arbitrary functions of u, v, v'.*

Property II. The two tangents at the node of the focal locus, associated with each element (u, v, v') by property I, are such that the one which has the direction of the given element bisects the angle between the other and a certain direction $\omega(u, v)$ through the given point (this direction is that of the force vector in the case of a trajectory system).

The most general system of ∞^3 curves on a surface possessing properties I and II is defined by a differential equation of the form

$$(II) \qquad v''' = A + Bv'' + \frac{3}{v' - \omega} v''^2,$$

where A and B are arbitrary functions of u, v, v', and ω is an arbitrary function of u, v.

Property III. Through every point and in every direction through that point there passes one curve of the system which hyperosculates its corresponding geodesic circle of curvature. The locus of the centers of geodesic curvature of the ∞^1 hyperosculating trajectories which pass through a point is a conic passing through the point in the direction $\omega(u, v)$ of property II.

The most general system of ∞^3 curves on a surface possessing properties I, II, III is defined by a differential equation of the form

$$(III) \qquad (\omega - v')H' = H(\gamma_0 + \gamma_1 v' + \gamma_2 v'^2 - 3v''),$$

involving four arbitrary functions, γ_0, γ_1, γ_2, ω of u, v, and where

$$(2) \quad H \equiv v'' - \frac{E_v}{2G} + \left(\frac{G_u}{G} - \frac{E_u}{2E}\right) v' + \left(\frac{G_v}{2G} - \frac{E_v}{E}\right) v'^2 + \frac{G_u}{2E} v'^3 = 0$$

is the differential equation of the geodesics on the surface, and

$$H' = dH/du.$$

Property IV. With each point O on the surface, property III associates a direction, viz., the tangent to the central locus or conic. The totality of all such directions on the surface defines a simple system of ∞^1 curves, which may be called the tangential lines (these are the lines of force in the trajectory system). The geodesic curvature of the tangential line through O is equal to 3 times the geodesic curvature of that hyperosculating curve which passes through O in the same direction.

The most general system of ∞^3 curves on a surface possessing properties I, II, III, IV is defined by a differential equation of the form (III) together with a condition on the functions γ_0, γ_1, γ_2, ω, i.e., by

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* Throughout this paper, subscripts refer to partial derivatives with respect to the indicated variable and primes refer to total derivatives with respect to u.

$$(IV)\quad \begin{cases} (\omega - v')H' = H(\gamma_0 + \gamma_1 v' + \gamma_2 v'^2 - 3v'') \\ \gamma_0 + \gamma_1\omega + \gamma_2\omega^2 = (\omega_u + \omega\omega_v) \\ -2\left[-\dfrac{E_v}{2G} + \left(\dfrac{G_u}{G} - \dfrac{E_u}{2E}\right)\omega + \left(\dfrac{G_v}{2G} - \dfrac{E_v}{E}\right)\omega^2 + \dfrac{G_u}{2E}\omega^3 \right]. \end{cases}$$

Property V. Construct any isothermal net on the surface. At any point O this net determines two orthogonal directions in which there pass two isothermal curves of the net and two hyperosculating curves of property *III*. If ρ_1, ρ_2, R_1, R_2 are the radii of geodesic curvature of these four curves, s_1, s_2, the arc lengths along the isothermal curves, and ω, the tangent of the angle between the tangent line to the conic of property *III* and the isothermal curve with arc s_2, then as we move along the surface from O, these quantities vary so as to satisfy the relation

$$\frac{\partial}{\partial s_2}\left(\frac{1}{\kappa_1}\right) - \frac{\partial}{\partial s_1}\left(\frac{1}{\kappa_2}\right) - \frac{1}{\rho_1\kappa_1} + \frac{1}{\rho_2\kappa_2} - \frac{\partial^2}{\partial s_1 \partial s_2}(\log \omega) = 0,$$

where

$$\frac{1}{\kappa_1} = \omega\left(\frac{1}{\rho_1} - \frac{3}{R_1}\right); \qquad \frac{1}{\kappa_2} = \frac{1}{\omega}\left(\frac{1}{\rho_2} - \frac{3}{R_2}\right).$$

The most general system of ∞^3 curves on a surface possessing properties *I, II, III, IV, V* is defined by a differential equation of the form

$$(V)\qquad \left(\frac{\psi}{G} - \frac{\phi}{E}v'\right)H' = H\left(\delta_0 + \delta_1 v' + \delta_2 v'^2 - 3\frac{\phi}{E}v''\right),$$

where

$$(3)\quad \begin{cases} \delta_0 = \dfrac{\psi E_u + \phi E_v}{EG} + \dfrac{G\psi_u - \psi G_u}{G^2}, \quad \delta_2 = \dfrac{\phi E_v - E\phi_v}{E^2} - \dfrac{\psi G_u + \phi G_v}{EG}, \\ \delta_1 = \dfrac{2\psi E_v - 2\phi G_u}{EG} + \dfrac{G\psi_v - \psi G_v}{G^2} + \dfrac{\phi E_v - E\phi_u}{E^2}, \end{cases}$$

and where ϕ and ψ are arbitrary functions of u, v.

Equation (V) is the differential equation of the system of ∞^3 trajectories generated by a point moving on a surface under any positional field of force, the components of which along the tangent lines to the parameter curves are $\phi/\sqrt{E}$ and $\psi/\sqrt{G}$.* These determine a direction

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* The trajectories on a surface are in general defined by the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \phi, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \psi,$$

where T is the kinetic energy, $\dot{u} = du/dt$, $\dot{v} = dv/dt$, $\phi = Xx_u + Yy_u + Zz_u$, $\psi = Xx_v + Yy_v + Zz_v$, X, Y, Z being the components of the force along the coördinate axes. See "Proceedings," § 2.

$\omega = dv/du = E\psi/G\phi$ along the surface, which we shall call the direction of the force vector.

2. Arbitrary point transformations. Consider any real representation of a surface S on a surface $\bar{S}$, whereby a one-to-one correspondence is established between the points of the two surfaces. In such an arbitrary point transformation there is at least one real orthogonal set of curves on S which corresponds to a real orthogonal set on $\bar{S}$.* If we choose the curves of these two orthogonal sets as parameter curves, and corresponding curves are assigned the same parameter value u or v, then corresponding points will have the same curvilinear coördinates u, v, and the elements of length are given by

$$(4) \qquad ds^2 = Edu^2 + Gdv^2, \qquad d\bar{s}^2 = \bar{E}du^2 + \bar{G}dv^2.$$

For the surface $\bar{S}$ we may set up the five types of differential equations $(\bar{I}), \cdots, (\bar{V})$, by merely writing $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{\omega}$, $\bar{\gamma}_0$, $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{E}$, $\bar{G}$, $\bar{\phi}$, $\bar{\psi}$, for the corresponding letters in equations $(I), \cdots, (V)$. Now, if (I) is to be converted into $(\bar{I})$, it is necessary and sufficient that

$$(5) \qquad \bar{A} = A, \qquad \bar{B} = B, \qquad \bar{C} = C.$$

But since these coefficients are arbitrary functions of u, v, v', conditions (5) can always be satisfied. Similarly, if (II) is to be converted into $(\overline{II})$, it is necessary and sufficient that

$$(6) \qquad \bar{A} = A, \qquad \bar{B} = B, \qquad \bar{\omega} = \omega.$$

But since the coefficients are arbitrary functions, conditions (6) can always be satisfied. Hence we may state

THEOREM 1. *An arbitrary point transformation will convert any system of curves with property I or any system of curves with properties I and II on S into a like system on* $\bar{S}$.

3. Geodesic transformations. It is evident that an arbitrary point transformation will not convert (III) into $(\overline{III})$. To find the most general point transformation that will make this conversion, we note that such a transformation must convert the part common to all systems of type (III) into the part common to all systems of type $(\overline{III})$. It is evident that $H = 0$ and $\bar{H} = 0$ satisfy (III) and $(\overline{III})$ respectively, and that the curves defined by these equations are the only proper curves satisfying all equations of these types. But the curves $H = 0$ and $\bar{H} = 0$ are the geodesics on S and $\bar{S}$ respectively. Hence the desired transformation must convert the geodesics on S into the geodesics on $\bar{S}$. Such a transformation is called a geodesic transformation. In order that the geodesics on S and $\overline{S}$ should correspond, it is necessary and sufficient that the differential equations $H = 0$ and $\overline{H} = 0$ be identical. We are thus led to the equations of condition

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* See G. Scheffers, Anwendung der Differential- und Integral-Rechnung auf Geometrie, vol. 2, p. 96.

$$(7) \qquad \left\{ \begin{array}{ll} -\dfrac{\overline{E}_v}{2\overline{G}} = -\dfrac{E_v}{2G}, & \dfrac{\overline{G}_u}{\overline{G}} - \dfrac{\overline{E}_u}{2\overline{E}} = \dfrac{G_u}{G} - \dfrac{E_u}{2E}, \\ \dfrac{\overline{G}_u}{2\overline{E}} = \dfrac{G_u}{2E}, & \dfrac{\overline{G}_v}{2\overline{G}} - \dfrac{\overline{E}_v}{\overline{E}} = \dfrac{G_v}{2G} - \dfrac{E_v}{E}, \end{array} \right.$$

which must hold identically. We note that this transformation will necessarily convert H into $\overline{H}$ and H' into $\overline{H}'$.

Now, if a geodesic transformation is to convert (*III*) into ($\overline{III}$), it is necessary and sufficient that

$$(8) \qquad \overline{\omega} = \omega, \qquad \overline{\gamma}_0 = \gamma_0, \qquad \overline{\gamma}_1 = \gamma_1, \qquad \overline{\gamma}_2 = \gamma_2.$$

Since these coefficients are arbitrary functions of u, v, conditions (8) can always be satisfied. Hence, we have

THEOREM 2. *The most general point transformation that converts any system of curves with properties I, II, III on S into a like system on $\overline{S}$, is the geodesic transformation.*

Now, equation (*IV*) involves the quantities E and G, which are not arbitrary. But it is evident that conditions (7) and (8) are both necessary and sufficient in order that (*IV*) be converted into ($\overline{IV}$). Hence, we have

THEOREM 3. *The most general point transformation that converts any system with properties I, II, III, IV on S into a like system on $\overline{S}$ is the geodesic transformation.*

Since type (*V*) is a special form of type (*III*), the most general point transformation that would convert any system of dynamical trajectories on S into a like system on $\overline{S}$ must be the geodesic transformation. We shall now examine whether every such transformation actually does convert a system of dynamical trajectories into a like system.

Dini first proved the theorem that if the real representation of a surface S on another $\overline{S}$ is geodesic, three cases are possible: (*i*) $\overline{S}$ may be obtained from S by a pure bending, i.e., $\overline{S}$ is applicable on S; (*ii*) $\overline{S}$ may be obtained from S by a similitude transformation with or without a bending; (*iii*) S and $\overline{S}$ are Liouville surfaces.* Let us apply these results to type (*V*).

(*i*) $\overline{S}$ is applicable on S; then

$$(9) \qquad \overline{E} = E, \qquad \overline{G} = G.$$

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* See Scheffers, ibid., vol. 2, p. 420.

For (V) to be converted into $(\overline{V})$, corresponding coefficients must be equal. This leads to

$$\frac{\overline{\psi}}{\psi} = \frac{\overline{\phi}}{\phi} = \frac{\overline{\delta}_0}{\delta_0} = \frac{\overline{\delta}_1}{\delta_1} = \frac{\overline{\delta}_2}{\delta_2},$$

and these conditions reduce to

$$\frac{\overline{\psi}}{\psi} = \frac{\overline{\phi}}{\phi}, \qquad \frac{\overline{\psi}_u}{\overline{\psi}} = \frac{\psi_u}{\psi}, \qquad \frac{\overline{\phi}_v}{\overline{\phi}} = \frac{\phi_v}{\phi}, \qquad \frac{\overline{\psi}_v}{\overline{\psi}} - \frac{\overline{\phi}_u}{\overline{\psi}} = \frac{\psi_v}{\psi} - \frac{\phi_u}{\psi},$$

and hence,

$$\overline{\psi} = c\psi, \qquad \overline{\phi} = c\phi, \tag{10}$$

where c is an arbitrary constant. We may take $c = 1$, since ϕ, ψ and $c\phi$, $c\psi$ define the same field of force. Conditions (10) can always be satisfied since $\overline{\phi}$, $\overline{\psi}$ are arbitrary functions of u, v.

(*ii*) $\overline{S}$ is obtained from S by a similitude transformation, i.e., the rectangular coördinates of corresponding points on the two surfaces are connected by the relations

$$\overline{x} = kx, \qquad \overline{y} = ky, \qquad \overline{z} = kz, \qquad (k = \text{constant});$$

then

$$\overline{E} = k^2E, \qquad \overline{G} = k^2G. \tag{11}$$

Again, if (V) is to be converted into $(\overline{V})$, then, as in case (*i*), $\overline{\psi} = \psi$, $\overline{\phi} = \phi$. Hence, we have

THEOREM 4. *Any system of dynamical trajectories on S is converted by a geodesic transformation into a system of dynamical trajectories on $\overline{S}$, provided $\overline{S}$ is applicable on S or $\overline{S}$ is applicable on a surface which can be obtained from S by a similitude transformation. The components of force on S and $\overline{S}$ are the same functions of the coördinates u, v.*

(*iii*) S and $\overline{S}$ are Liouville surfaces. A Liouville surface is characterized by the fact that the element of length may be reduced to the form

$$ds^2 = (U + V)(du^2 + dv^2), \tag{12}$$

where U is a function of u only and V a function of v only.* The corresponding surface $\overline{S}$ has for element of length

$$d\overline{s}^2 = -\left(\frac{1}{U} + \frac{1}{V}\right)\left(\frac{du^2}{U} - \frac{dv^2}{V}\right). \tag{13}$$

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* To the Liouville surfaces belong, among others, the surfaces of constant curvature, the surfaces of revolution, and the quadrics. When the element of length can be written in the form (12), the parameter curves are geodesic ellipses and hyperbolas. Cf. L. P. Eisenhart, Differential Geometry, p. 215. For Liouville surfaces the finite equation of the geodesics may be found by simple quadratures. For a discussion of surfaces of this type, see Darboux, Leçons sur la Théorie Générale des Surfaces, vol. II, Chap. IX.

It is easily seen that $d\bar{s}^2$ has the Liouville form, for by a change of parameters

$$\bar{u} = \int \frac{du}{\sqrt{U}}, \qquad \bar{v} = \int \frac{dv}{\sqrt{-V}},$$

(13) takes the form

$$d\bar{s}^2 = -\left(\frac{1}{U} + \frac{1}{V}\right)(d\bar{u}^2 + d\bar{v}^2) = (\bar{U} + \bar{V})(d\bar{u}^2 + d\bar{v}^2).$$

Now, on the surface S defined by (12), type (V) takes the form

$$(\psi - \phi v')H' = H(a_1 + a_2 v' + a_3 v'^2 - 3\phi v''), \tag{14}$$

where

$$a_1 = \psi_u + \frac{\phi V_v}{U + V}, \qquad a_2 = \psi_v - \phi_u + \frac{\psi V_v - \phi U_u}{U + V},$$
$$a_3 = -\phi_v - \frac{\psi U_u}{U + V}.$$

On the surface $\bar{S}$ defined by (13), type $(\bar{V})$ takes the form

$$\left(-\frac{V\bar{\psi}}{U} - \bar{\phi}v'\right)H' = H(b_1 + b_2 v' + b_3 v'^2 - 3\bar{\phi}v''), \tag{15}$$

where

$$b_1 = \frac{V(\bar{\psi}U_u - U\bar{\psi}_u)}{U^2} + \frac{\bar{\phi}V_v}{U + V}, \quad b_3 = \frac{\bar{\phi}V_v - V\bar{\phi}_v}{V} + \frac{\bar{\psi}VU_u}{U(U + V)},$$
$$b_2 = -\frac{V\bar{\psi}_v}{U} - \bar{\phi}_u - \frac{\bar{\psi}VV_v + \bar{\phi}UU_u}{U(U + V)}.$$

Here ϕ, ψ and $\bar{\phi}$, $\bar{\psi}$ are arbitrary functions of u, v determining the force vectors on S and $\bar{S}$ respectively. Let us now see whether by a proper choice of $\bar{\phi}$ and $\bar{\psi}$, equation (14) may be converted into equation (15). For such conversion, corresponding coefficients must be equal, i.e.,

$$\frac{\psi}{\phi} = -\frac{V\bar{\psi}}{U\bar{\phi}}, \qquad \frac{a_1}{\phi} = \frac{b_1}{\bar{\phi}}, \qquad \frac{a_2}{\phi} = \frac{b_2}{\bar{\phi}}, \qquad \frac{a_3}{\phi} = \frac{b_3}{\bar{\phi}} \tag{16}$$

Combining the first two conditions, we find

$$\frac{\bar{\psi}_u}{\bar{\psi}} = \frac{U_u}{U} + \frac{\psi_u}{\psi}.$$

Hence

$$\bar{\psi} = V_1 U \psi, \qquad \bar{\phi} = -V_1 V \phi, \tag{17}$$

where V_1 is an arbitrary function of v only. Combining the first and third condition, we find

$$\frac{\bar{\phi}_v}{\bar{\phi}} = \frac{V_v}{V} + \frac{\phi_v}{\phi},$$

and substituting the value of $\bar{\phi}$ found in (17), this becomes

$$V_1 = \text{constant} = c.$$

Hence,

$$\bar{\psi} = cU\psi, \qquad \bar{\phi} = -\,cV\phi. \tag{18}$$

By substitution we find that these relations satisfy the fourth condition. As in case (*i*) we may take $c = 1$, so that

$$\bar{\psi} = U\psi, \qquad \bar{\phi} = -\,V\phi. \tag{19}$$

These conditions can always be satisfied, since $\bar{\phi}$, $\bar{\psi}$ are arbitrary functions of u, v. Hence, we have

THEOREM 5. *A system of dynamical trajectories on a Liouville surface S is converted by a geodesic transformation into a like system on the corresponding Liouville surface $\bar{S}$. The components of force ϕ, ψ on S and the components $\bar{\phi}$, $\bar{\psi}$ on $\bar{S}$ are related by $\bar{\psi} = U\psi$, $\bar{\phi} = -\,V\phi$.*

Finally, combining Theorems 4 and 5, we may state

THEOREM 6. *Any geodesic transformation of a surface S into a surface $\bar{S}$ will convert any system of dynamical trajectories on S into a like system on $\bar{S}$; a geodesic transformation is the most general point transformation that makes this conversion possible.*

4. Conservative forces. If the field of force is conservative, then $\psi_u = \phi_v$. This condition is characterized geometrically by the fact that the conic of property *III* becomes a rectangular hyperbola.* The question arises: is a system of dynamical trajectories in a conservative field of force converted by a geodesic transformation into a like system? This may be answered in the affirmative for cases (*i*) and (*ii*) in our discussion of geodesic transformations, for, by Theorem 4, $\bar{\psi} = \psi$, $\bar{\phi} = \phi$; hence, if $\psi_u = \phi_v$, then $\bar{\psi}_u = \bar{\phi}_v$. But for case (*iii*), where S and $\bar{S}$ are Liouville surfaces, we have, by Theorem 5, $\bar{\psi} = U\psi$, $\bar{\phi} = -\,V\phi$. Now if $\psi_u = \phi_v$ and $\bar{\psi}_u = \bar{\phi}_v$, we must have

$$(U + V)\phi_v + U_u\psi + V_v\phi = 0. \tag{20}$$

Now, if W is the work function (negative potential), then

$$\phi = W_u, \qquad \psi = W_v$$

and (20) becomes the Laplacian equation

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* "Proceedings," § 9.

$$(U + V)W_{uv} + V_v W_u + U_u W_v = 0, \tag{21}$$

or

$$[(U + V)W]_{uv} = 0, \tag{22}$$

the solution of which is

$$(U + V)W = U_1 + V_1, \tag{23}$$

where U_1 is an arbitrary function of u alone, and V_1 is an arbitrary function of v alone. Hence, we may state

Theorem 7. *A geodesic transformation will convert a system of dynamical trajectories in a conservative field of force on any surface S into a like system on $\bar{S}$, provided $\bar{S}$ is applicable on S, or $\bar{S}$ is applicable on a surface which may be obtained from S by a similitude transformation. A geodesic transformation will convert a system of dynamical trajectories in a conservative field of force on a Liouville surface S into a like system on the corresponding Liouville surface $\bar{S}$ only if W, the work function, has the form*

$$W = (U_1 + V_1)/(U + V),$$

where U_1 and V_1 are arbitrary functions of u and v respectively.

From equation (23) it is evident that there are no Liouville surfaces for which the transformation is possible if W is to be an arbitrary point function.

5. "N" systems. If the field of force is conservative, we may study the transformations of certain types of ∞^3 curves on a surface other than dynamical trajectories. These systems, termed "*n*" systems,* are characterized by the differential equation

$$(V_n) \quad \left\{ \begin{aligned} &\left(\frac{\psi}{G} - \frac{\phi}{E}v'\right)(E + Gv'^2)H' \\ &\qquad = H\Big\{ (E + Gv'^2)\left(\epsilon_0 + \epsilon_1 v' + \epsilon_2 v'^2 - \frac{3\phi}{E}v''\right) \\ &\qquad + \frac{(n-2)v'}{2}(\zeta_0 + \zeta_1 v' + \zeta_2 v'^2) - (n-2)(\phi + \psi v')v''\Big\} \end{aligned} \right.$$

where

$$\epsilon_0 = \frac{2\psi E_u + n\phi E_v}{2EG} + \frac{G\psi_u - \psi G_u}{G^2}, \qquad \epsilon_2 = -\frac{n\psi G_u + 2\phi G_v}{2EG} + \frac{\phi E_v - E\phi_v}{E^2},$$

$$\epsilon_1 = \frac{(2+n)(\psi E_v - \phi G_u)}{2EG} + \frac{G\psi_v - \psi G_v}{G^2} + \frac{\phi E_u - E\phi_u}{E^2}$$

and

$$\zeta_0 = \frac{\phi E_u}{E} - \frac{\phi G_u}{G}, \quad \zeta_2 = \frac{\psi E_v}{E} - \frac{\psi G_v}{G}, \quad \zeta_1 = \frac{\psi E_u}{E} - \frac{\psi G_u}{G} + \frac{\phi E_v}{E} - \frac{\phi G_v}{G}$$

and where $\phi_v = \psi_u$.

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* "Proceedings," § 10.

An " n " system is a system of dynamical trajectories when $n = 2$, velocity curves when $n = 0$, brachistochrones when $n = -2$, catenaries when $n = 1$. Even if the field is not conservative, equation (V_n) may still be said to define " n " systems—dynamical trajectories, velocity curves, pseudo-brachistochrones, pseudo-catenaries.

In " Proceedings " we have given five geometric properties—I_n, II_n, III_n, IV_n, V_n—which completely characterize such " n " systems. These properties are analogous to properties $I, \cdots, V$ which characterize the trajectories under any positional field of force. In fact, I and I_n are the same; to get II_n we replace the equal angles in II by two angles whose tangents are in the ratio $n + 1 : 3$; III and III_n are the same—for the conservative case the conic must be a rectangular hyperbola; to get IV_n and V_n, we replace the multiple 3 in IV and V by $n + 1$. Each of these properties together with the preceding may be characterized by differential equations $(I_n), \cdots, (V_n)$ similar to equations $(I), \cdots, (V)$.

Equation (V_n) is given above. The others are

$$(I_n) \qquad v''' = A + Bv'' + Cv''^2,$$

$$(II_n) \qquad v''' = A + Bv'' + \frac{1}{\omega - v'}\left[\frac{(2-n)(E + G\omega v')}{E + Gv'^2} - 3\right]v''^2,$$

$$(III_n) \qquad \begin{aligned}(\omega - v')H' = H\Bigg\{\gamma_0 + \gamma_1 v' + \gamma_2 v'^2 \\ + \left[\frac{(2-n)(E + G\omega v')}{E + Gv'^2} - 3\right]v''\Bigg\},\end{aligned}$$

$$(IV_n) \qquad \left\{\begin{aligned}&(\omega - v')H' = H\Bigg\{\gamma_0 + \gamma_1 v' + \gamma_2 v'^2 \\ &\qquad\qquad + \left[\frac{(2-n)(E + G\omega v')}{E + Gv'^2} - 3\right]v''\Bigg\}, \\ &\gamma_0 + \gamma_1\omega + \gamma_2\omega^2 = (\omega_u + \omega\omega_v) - n\Bigg\{-\frac{E_v}{2G} \\ &\qquad\qquad + \left(\frac{G_u}{G} - \frac{E_u}{2E}\right)\omega + \left(\frac{G_v}{2G} - \frac{E_v}{E}\right)\omega^2 + \frac{G_u}{2E}\omega^3\Bigg\}.\end{aligned}\right.$$

The quantities $A, B, C, \omega, \gamma_0, \gamma_1, \gamma_2, \psi, \phi, H, H'$ have the same significance as in the previous discussion.

A study of equations $(I_n), \cdots, (V_n)$ similar to the study made of equations $(I), \cdots, (V)$ leads to the following results:

1. Type (I_n) alone is conserved under an *arbitrary* point transformation.

2. In order that (II_n) be converted into $(\overline{II}_n)$, it is necessary and sufficient that $\overline{E}/E = \overline{G}/G$, i.e., the transformation must be conformal.

3. In order that (III_n) be converted into $(\overline{III}_n)$, the transformation must be *geodesic*, but not every such transformation makes the desired conversion. Under case (*i*), $\overline{E} = E, \overline{G} = G$; and under case (*ii*), $\overline{E} = k^2E$, $\overline{G} = k^2G$. These types of geodesic transformation evidently convert (III_n) into $(\overline{III}_n)$. But under case (*iii*), i.e., on Liouville surfaces, where $E = G = U + V$, and

$$\overline{E} = -\left(\frac{1}{U}+\frac{1}{V}\right)\frac{1}{U}, \qquad \overline{G} = \left(\frac{1}{U}+\frac{1}{V}\right)\frac{1}{V},$$

(III_n) is converted into $(\overline{III}_n)$ if, and only if, $U + V = 0$, i.e., the element of length ds vanishes over the entire surface. Hence, a geodesic transformation on a Liouville surface will not make the desired conversion.

4. A geodesic transformation of types (*i*) and (*ii*) will convert (IV_n) into $(\overline{IV}_n)$ and (V_n) into $(\overline{V}_n)$ with $\overline{\psi} = \psi$, $\overline{\phi} = \phi$.

Of course, the more general results of the earlier discussion hold for the case $n = 2$, i.e., dynamical trajectories.

We may now state

THEOREM 8. *An arbitrary point transformation will convert any system of curves with property I_n on S into a like system on $\overline{S}$. The most general point transformation that will convert any system of curves with properties I_n, II_n on S into a like system on $\overline{S}$ is the conformal transformation. The most general point transformation that will convert systems of curves with properties I_n, II_n, III_n, or I_n, II_n, III_n, IV_n, or I_n, II_n, III_n, IV_n, V_n (i.e., an "n" system) on S into a like system on $\overline{S}$ is the geodesic transformation under which $\overline{S}$ is applicable on S or $\overline{S}$ is applicable on a surface which can be obtained from S by a similitude transformation. Conservative fields are converted into conservative fields.*

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